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Note on a Casimir energy calculation for a purely dielectric cylinder by mode summation

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Abstract

We comment on a recent calculation of the zero-point energy for a dilute and infinitely long cylinder of purely-dielectric material. The vanishing result predicted by integration of van der Waals potentials is obtained.

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The Casimir effect is a change in the electromagnetic vacuum fluctuations brought about by the presence of boundaries. Particularly, cylindrical surfaces limiting dielectric media were considered in [1]. One of the first versions of that paper inspired an unpublished calculation, by Romeo, of the van der Waals energy for a purely dielectric cylinder in the dilute-dielectric approximation, which yielded a null result. That calculation found a tribune in appendix B of the final version of [1] and, eventually, unpublished work by Milonni and [2] by Barton provided independent confirmations.

This finding aroused curiosity about the corresponding Casimir energy, which would have to show the predicted equality between both quantities [3] and, therefore, was expected to vanish similarly. The divergences of this problem were studied through its heat kernel coefficents in [4], and the expected vanishing was first verified in [5], where the Casimir pressure was obtained from the expectation value of the stress–energy tensor using Green's functions. Next, a calculation of the Casimir energy based on the mode summation method [6] was completed. The present paper offers a comment on that work.

Let J_m, H_m denote the Bessel and Hankel functions (for $y > 0, H_m(y) \equiv H_m^{(1)}(y)$). Given an infinitely long cylinder of radius a, oriented along the z-axis, with permittivity and permeability (ε_1, μ_1), surrounded by a medium with permittivity and permeability (ε_2, μ_2), the eigenfrequencies ω of the Maxwell equations with the adequate boundary conditions are the solutions of

$$f_m(k_z, \omega) = 0, \qquad m \in \mathbb{Z}, \qquad k_z \in \mathbb{R},$$
$$f_m(k_z, \omega) \equiv \frac{1}{\Delta^2}\left[\Delta_m^{\mathrm{TE}}(x, y)\Delta_m^{\mathrm{TM}}(x, y) - m^2\frac{a^4\omega^2k_z^2}{x^2y^2}(\varepsilon_1\mu_1 - \varepsilon_2\mu_2)^2J_m^2(x)H_m^2(y)\right] \tag{1}$$

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(see [7, 1]), where

$$\begin{aligned}
&\Delta = -\frac{2\mathrm{i}}{\pi}, \\
&\Delta_m^{\mathrm{TE}}(x, y) = \mu_1 y J_m'(x) H_m(y) - \mu_2 x J_m(x) H_m'(y), \\
&\Delta_m^{\mathrm{TM}}(x, y) = \varepsilon_1 y J_m'(x) H_m(y) - \varepsilon_2 x J_m(x) H_m'(y), \\
&x = \lambda_1 a, \qquad y = \lambda_2 a, \qquad \lambda_i^2 = \varepsilon_i \mu_i \omega^2 - k_z, \quad i = 1, 2.
\end{aligned} \tag{2}$$

The m index is the azimuthal quantum number, k_z is the momentum along the cylinder axis, and p labels the zeros of $f_m(k_z, \omega)$. In fact $f_m = -\Delta^{-2}\Xi$, Ξ being the same object as in [5] and Δ^{-2} a factor introduced for convenience. The velocities of light in each media are $c_i = (\varepsilon_i \mu_i)^{-1/2}$, $i = 1, 2$.

If medium 1 is purely dielectric and medium 2 is vacuum, $\varepsilon_1 = \varepsilon$, $\mu_1 = 1$, $\varepsilon_2 = \mu_2 = 1$ (obviously, $c_2 = 1$). Further,

$$\omega = a^{-1}(y^2 + \widehat{k}^2)^{1/2}, \qquad x^2 = y^2 + (\varepsilon - 1)(y^2 + \widehat{k}^2), \qquad \widehat{k} \equiv k_z a. \tag{3}$$

The Casimir energy per unit length stems from the mode sum

$$\mathcal{E}_C = \frac{1}{2}\hbar \int_{-\infty}^{\infty} \frac{\mathrm{d}k_z}{2\pi} \sum_m \sum_p \omega_{m,p,k_z}, \tag{4}$$

which is divergent, and will be regularized appropriately (see below). Reference [4] tells us that, up through the order of $(\varepsilon - 1)^2$, there are no ambiguities, because the heat kernel coefficient which would multiply them is of $\mathcal{O}((\varepsilon - 1)^3)$. Thus, we may just set

$$\mathcal{E}_C(s) = \frac{\hbar}{2} \int_{-\infty}^{\infty} \frac{\mathrm{d}k_z}{2\pi} \sum_m \sum_p \omega_{m,p,k_z}^{-s} = \frac{\hbar}{2} a^{s-1} \int_{-\infty}^{\infty} \frac{\mathrm{d}\widehat{k}}{2\pi} \sum_m \sum_p \left(y_{m,p}^2 + \widehat{k}^2\right)^{-s/2}, \tag{5}$$

without any additional mass scale. $\mathcal{E}_C(s)$ is a function of the complex variable s, and our idea is to redefine (4) by analytic continuation of this function to $s = -1$, i.e.,

$$\mathcal{E}_C = \lim_{s \to -1} \mathcal{E}_C(s). \tag{6}$$

Once that $\widehat{k}, m$ have specific values, the sum over p is expressed as a contour integral in complex y plane:

$$\mathcal{E}_C(s) = \frac{\hbar}{2} a^{s-1} \int_{-\infty}^{\infty} \frac{\mathrm{d}\widehat{k}}{2\pi} \sum_{m=-\infty}^{\infty} \frac{s}{2\pi \mathrm{i}} \int_C \mathrm{d}y\, y(y^2 + \widehat{k}^2)^{-s/2-1} \ln f_m, \tag{7}$$

where C is a circuit enclosing all the y values corresponding to the positive zeros of f_m (the argument principle [8] derived from the residue theorem). When applying this method, one sometimes finds an asymptotic form $f_{m,\mathrm{as}}$ of f_m and then subtracts $\ln f_{m,\mathrm{as}}$ from $\ln f_m$ in the integrand. In fact, the factors introduced in (1) relative to the original f_m of [1] have the same effect as having divided that function by the leading part of $f_{m,\mathrm{as}}$.

At this point, the logarithm function of (7) is expanded in powers of $(\varepsilon - 1)$, taking y as an independent variable and x as a function of $y, \widehat{k}, \varepsilon$ (see (3)). Then,

$$\begin{aligned}
\ln f_m = {}& \left[L_{m1}^0(y) + L_{m1}^1(y)(y^2 + \widehat{k}^2)\right](\varepsilon - 1) + \left[L_{m2}^{00}(y) + L_{m2}^{10}(y)(y^2 + \widehat{k}^2)\right. \\
& \left. + L_{m2}^{20}(y)(y^2 + \widehat{k}^2)^2 + L_{m2}^{11}(y)(y^2 + \widehat{k}^2)\widehat{k}^2\right](\varepsilon - 1)^2 + \mathcal{O}((\varepsilon - 1)^3),
\end{aligned} \tag{8}$$

where

$$
\begin{aligned}
L^0_{m1}(y) &= \frac{1}{\Delta} y J'_m(y) H_m(y),\\
L^1_{m1}(y) &= \frac{1}{\Delta y} \Delta^{(1,0)}_m(y),\\
L^{00}_{m2}(y) &= -\frac{1}{2\Delta^2} y^2 J'^{\,2}_m(y) H^2_m(y),\\
L^{10}_{m2}(y) &= -\frac{1}{2\Delta^2}\left[\Delta^{(1,0)}_m(y) J'_m(y) H_m(y) + \frac{\Delta}{y}\left(J'_m(y) + y\left(1-\frac{m^2}{y^2}\right) J_m(y)\right) H_m(y)\right],\\
L^{20}_{m2}(y) &= L^{20A}_{m2}(y) + L^{20B}_{m2}(y), \qquad \begin{cases} L^{20A}_{m2}(y) = \dfrac{1}{4\Delta y^2}\left(\Delta^{(2,0)}_m(y) - \dfrac{1}{y}\Delta^{(1,0)}_m(y)\right),\\ L^{20B}_{m2}(y) = -\dfrac{1}{4\Delta^2 y^2}\left(\Delta^{(1,0)}_m(y)\right)^2, \end{cases}\\
L^{11}_{m2}(y) &= -\frac{m^2}{\Delta^2 y^4} J^2_m(y) H^2_m(y),
\end{aligned}
\tag{9}
$$

with

$$
\begin{aligned}
\Delta^{(1,0)}_m(y) &= -\frac{1}{y}[y^2 J'_m(y) H'_m(y) + (y^2 - m^2) J_m(y) H_m(y)] - (J_m(y) H_m(y))',\\
\Delta^{(2,0)}_m(y) &= \left(\Delta^{(1,0)}_m(y)\right)' - \left(1 - \frac{m^2+1}{y^2}\right)\Delta, \qquad \left(\Delta^{(1,0)}_m(y)\right)' \equiv \frac{\mathrm{d}}{\mathrm{d}y}\Delta^{(1,0)}_m(y).
\end{aligned}
\tag{10}
$$

Now, (8) is inserted into (7). The obtained expression involves integrals of the form

$$
I \equiv \int_{-\infty}^{\infty} \mathrm{d}\widehat{k} \int_C \mathrm{d}y\, y F(y)(y^2 + \widehat{k}^{\,2})^{-\alpha} \widehat{k}^{2\beta}, \tag{11}
$$

where C is the contour of (7) and F satisfies $F(-\mathrm{i}v) = F(\mathrm{i}v)$ for $v \in \mathbb{R}$, as well as having good asymptotic properties (the role of F is played by the L_m's of (9), (10)). Examining the $(y^2 + \widehat{k}^{\,2})$ powers in (7), (8), one sees that, in the required cases, $\alpha = s/2 + 1, s/2, s/2 - 1$, and $\beta = 0$ except for one integral with $\beta = 1$. Analytic continuation in s obviously amounts to analytic continuation in α. Following [6], the value of I is given by

$$
I = -2\mathrm{i}\mathrm{B}\left(\beta + \frac{1}{2}, 1 - \alpha\right) \sin(\pi\alpha) \int_0^{\infty} \mathrm{d}v\, v^{2-2\alpha+2\beta} F(\mathrm{i}v), \tag{12}
$$

where B denotes the Euler beta function (about the mathematical basis, see also [9, 10]). Note that for $s = -1$, i.e., $\alpha = 1/2, -1/2, -3/2$, and for $\beta = 0, 1$, the beta and sine functions are finite. Application of formula (12) to equations (7), (8) gives

$$
\mathcal{E}_C(s) = \mathcal{E}_{C1}(s)(\varepsilon - 1) + \mathcal{E}_{C2}(s)(\varepsilon - 1)^2 + \mathcal{O}((\varepsilon - 1)^3), \tag{13}
$$

where

$$
\begin{aligned}
&\mathcal{E}_{C1}(s) = \mathcal{E}^0_{C1}(s) + \mathcal{E}^1_{C1}(s),\\
&\begin{cases} \mathcal{E}^0_{C1}(s) = -\dfrac{\hbar}{2}\dfrac{s a^{s-1}}{2\pi^2}\mathrm{B}\left(\dfrac{1}{2}, -\dfrac{s}{2}\right)\sin\left(-\pi\dfrac{s}{2}\right)\displaystyle\sum_{m=-\infty}^{\infty}\int_0^{\infty}\mathrm{d}v\, v^{-s} L^0_{m1}(\mathrm{i}v),\\ \mathcal{E}^1_{C1}(s) = -\dfrac{\hbar}{2}\dfrac{s a^{s-1}}{2\pi^2}\mathrm{B}\left(\dfrac{1}{2}, 1-\dfrac{s}{2}\right)\sin\left(\pi\dfrac{s}{2}\right)\displaystyle\sum_{m=-\infty}^{\infty}\int_0^{\infty}\mathrm{d}v\, v^{2-s} L^1_{m1}(\mathrm{i}v), \end{cases}
\end{aligned}
\tag{14}
$$

and

$$\mathcal{E}_{C2}(s) = \mathcal{E}^{00}_{C2}(s) + \mathcal{E}^{10}_{C2}(s) + \mathcal{E}^{20A}_{C2}(s) + \mathcal{E}^{20B}_{C2}(s) + \mathcal{E}^{11}_{C2}(s),$$

$$\begin{cases} \mathcal{E}^{00}_{C2}(s) = -\dfrac{\hbar}{2}\dfrac{sa^{s-1}}{2\pi^2}\mathrm{B}\left(\dfrac{1}{2}, -\dfrac{s}{2}\right)\sin\left(-\pi\dfrac{s}{2}\right)\displaystyle\sum_{m=-\infty}^{\infty}\int_0^\infty \mathrm{d}v\, v^{-s} L^{00}_{m2}(\mathrm{i}v), \\ \mathcal{E}^{10}_{C2}(s) = -\dfrac{\hbar}{2}\dfrac{sa^{s-1}}{2\pi^2}\mathrm{B}\left(\dfrac{1}{2}, 1-\dfrac{s}{2}\right)\sin\left(\pi\dfrac{s}{2}\right)\displaystyle\sum_{m=-\infty}^{\infty}\int_0^\infty \mathrm{d}v\, v^{2-s} L^{10}_{m2}(\mathrm{i}v), \\ \mathcal{E}^{20A,B}_{C2}(s) = -\dfrac{\hbar}{2}\dfrac{sa^{s-1}}{2\pi^2}\mathrm{B}\left(\dfrac{1}{2}, 2-\dfrac{s}{2}\right)\sin\left(-\pi\dfrac{s}{2}\right)\displaystyle\sum_{m=-\infty}^{\infty}\int_0^\infty \mathrm{d}v\, v^{4-s} L^{20A,B}_{m2}(\mathrm{i}v), \\ \mathcal{E}^{11}_{C2}(s) = -\dfrac{\hbar}{2}\dfrac{sa^{s-1}}{2\pi^2}\mathrm{B}\left(\dfrac{3}{2}, 1-\dfrac{s}{2}\right)\sin\left(\pi\dfrac{s}{2}\right)\displaystyle\sum_{m=-\infty}^{\infty}\int_0^\infty \mathrm{d}v\, v^{4-s} L^{11}_{m2}(\mathrm{i}v). \end{cases} \tag{15}$$

With $\mathcal{E}^0_{C1}(s)$ taken from (14), and $L^0_{m1}(\mathrm{i}v)$ from (9), we arrive at

$$\mathcal{E}^0_{C1}(s) = -\frac{\hbar}{2}\frac{sa^{s-1}}{2\pi^2}\mathrm{B}\left(\frac{1}{2}, -\frac{s}{2}\right)\sin\left(-\pi\frac{s}{2}\right)\sum_{m=-\infty}^{\infty}\int_0^\infty \mathrm{d}v\, v^{1-s} I'_m(v)K_m(v). \tag{16}$$

The beta and sine functions are already finite at $s = -1$, and the integral will be reexpressed by introducing the factor $1 = -vW[I_m(v), K_m(v)] = -v[I_m(v)K'_m(v) - I'_m(v)K_m(v)]$ for every m:

$$\int_0^\infty \mathrm{d}v\, v^{1-s}\sum_{m=-\infty}^{\infty} I'_m(v)K_m(v) = -\int_0^\infty \mathrm{d}v\, v^{2-s}\sum_{m=-\infty}^{\infty} I_m(v)I'_m(v)K_m(v)K'_m(v) + \int_0^\infty \mathrm{d}v\, v^{2-s}\sum_{m=-\infty}^{\infty} {I'_m}^2(v)K^2_m(v). \tag{17}$$

The summations over m will be performed by taking advantage of the addition theorem for the modified Bessel functions:

$$\sum_{m=-\infty}^{\infty} I_m(kr)K_m(k\rho)\,\mathrm{e}^{\mathrm{i}m\phi} = K_0(kR(r,\rho,\phi)) \qquad R(r,\rho,\phi) = \sqrt{r^2+\rho^2-2r\rho\cos\phi}, \qquad \rho > r. \tag{18}$$

Suitable manipulations of this identity ([5, 6, 11, 12]) yield

$$\int_0^\infty \mathrm{d}v\, v^{2-s}\sum_{m=-\infty}^{\infty} {I'_m}^2(v)K^2_m(v) =$$

$$\int_0^\infty \mathrm{d}v\, v^{2-s}\sum_{m=-\infty}^{\infty} {K'_m}^2(v)I^2_m(v) =$$

$$\int_0^\infty \mathrm{d}v\, v^{2-s}\sum_{m=-\infty}^{\infty} I_m(v)I'_m(v)K_m(v)K'_m(v) = \frac{1}{8\pi^{1/2}}\frac{\Gamma\left(\frac{5-s}{2}\right)\Gamma^2\left(\frac{3-s}{2}\right)\Gamma\left(\frac{1-s}{2}\right)}{\Gamma(3-s)}\frac{\Gamma\left(\frac{s}{2}\right)}{\Gamma\left(\frac{s+1}{2}\right)}$$

$$\int_0^\infty \mathrm{d}v\, v^{2-s}\sum_{m=-\infty}^{\infty} m^2 I_m(v)I'_m(v)K_m(v)K'_m(v) = \frac{1}{16\pi^{1/2}}\frac{\Gamma^4\left(\frac{5-s}{2}\right)}{\Gamma(5-s)}\frac{\Gamma\left(\frac{s-2}{2}\right)}{\Gamma\left(\frac{s+1}{2}\right)}$$

$$\int_0^\infty \mathrm{d}v\, v^{4-s} \sum_{m=-\infty}^{\infty} I_m'^{\,2}(v) K_m'^{\,2}(v) = \frac{1}{8\pi^{1/2}} \left[\frac{\Gamma^4\left(\frac{5-s}{2}\right)}{\Gamma(5-s)} \frac{\Gamma\left(\frac{s}{2}\right)}{\Gamma\left(\frac{s+1}{2}\right)} + \frac{\Gamma^2\left(\frac{5-s}{2}\right)\Gamma^2\left(\frac{3-s}{2}\right)}{\Gamma(4-s)} \frac{\Gamma\left(\frac{s-2}{2}\right)}{\Gamma\left(\frac{s-1}{2}\right)} + \frac{1}{4} \frac{\Gamma\left(\frac{5-s}{2}\right)\Gamma^2\left(\frac{3-s}{2}\right)\Gamma\left(\frac{1-s}{2}\right)}{\Gamma(3-s)} \frac{\Gamma\left(\frac{s-4}{2}\right)}{\Gamma\left(\frac{s-3}{2}\right)} \right]$$

$$\int_0^\infty \mathrm{d}v\, v^{4-s} \sum_{m=-\infty}^{\infty} I_m^2(v) K_m^2(v) = \frac{1}{8\pi^{1/2}} \frac{\Gamma^4\left(\frac{5-s}{2}\right)}{\Gamma(5-s)} \frac{\Gamma\left(\frac{s-4}{2}\right)}{\Gamma\left(\frac{s-3}{2}\right)}$$

$$\int_0^\infty \mathrm{d}v\, v^{2-s} \sum_{m=-\infty}^{\infty} m^2 I_m^2(v) K_m^2(v) = \frac{1}{16\pi^{1/2}} \frac{\Gamma\left(\frac{7-s}{2}\right)\Gamma^2\left(\frac{5-s}{2}\right)\Gamma\left(\frac{3-s}{2}\right)}{\Gamma(5-s)} \frac{\Gamma\left(\frac{s-4}{2}\right)}{\Gamma\left(\frac{s-1}{2}\right)}$$

$$\int_0^\infty \mathrm{d}v\, v^{-s} \sum_{m=-\infty}^{\infty} m^4 I_m^2(v) K_m^2(v) = \frac{1}{8\pi^{1/2}} \left[\frac{3}{4} \frac{\Gamma^4\left(\frac{5-s}{2}\right)}{\Gamma(5-s)} \frac{\Gamma\left(\frac{s-4}{2}\right)}{\Gamma\left(\frac{s+1}{2}\right)} + \frac{1}{2} \frac{\Gamma^2\left(\frac{5-s}{2}\right)\Gamma^2\left(\frac{3-s}{2}\right)}{\Gamma(4-s)} \frac{\Gamma\left(\frac{s-4}{2}\right)}{\Gamma\left(\frac{s-1}{2}\right)} + \frac{1}{4} \frac{\Gamma\left(\frac{5-s}{2}\right)\Gamma^2\left(\frac{3-s}{2}\right)\Gamma\left(\frac{1-s}{2}\right)}{\Gamma(3-s)} \frac{\Gamma\left(\frac{s-4}{2}\right)}{\Gamma\left(\frac{s-3}{2}\right)} \right]$$

$$\int_0^\infty \mathrm{d}v\, v^{3-s} \sum_{m=-\infty}^{\infty} I_m'^{\,2}(v) K_m(v) K_m'(v) =$$

$$\int_0^\infty \mathrm{d}v\, v^{3-s} \sum_{m=-\infty}^{\infty} K_m'^{\,2}(v) I_m(v) I_m'(v) = -\frac{1}{8\pi^{1/2}} \left[\frac{\Gamma^2\left(\frac{5-s}{2}\right)\Gamma^2\left(\frac{3-s}{2}\right)}{\Gamma(4-s)} \frac{\Gamma\left(\frac{s}{2}\right)}{\Gamma\left(\frac{s+1}{2}\right)} + \frac{1}{2} \frac{\Gamma\left(\frac{5-s}{2}\right)\Gamma^2\left(\frac{3-s}{2}\right)\Gamma\left(\frac{1-s}{2}\right)}{\Gamma(3-s)} \frac{\Gamma\left(\frac{s-2}{2}\right)}{\Gamma\left(\frac{s-1}{2}\right)} \right]$$

$$\int_0^\infty \mathrm{d}v\, v^{3-s} \sum_{m=-\infty}^{\infty} I_m^2(v) K_m(v) K_m'(v) =$$

$$\int_0^\infty \mathrm{d}v\, v^{3-s} \sum_{m=-\infty}^{\infty} K_m^2(v) I_m(v) I_m'(v) = -\frac{1}{8\pi^{1/2}} \frac{\Gamma^2\left(\frac{5-s}{2}\right)\Gamma^2\left(\frac{3-s}{2}\right)}{\Gamma(4-s)} \frac{\Gamma\left(\frac{s-2}{2}\right)}{\Gamma\left(\frac{s-1}{2}\right)}$$

$$\int_0^\infty \mathrm{d}v\, v^{1-s} \sum_{m=-\infty}^{\infty} m^2 I_m^2(v) K_m(v) K_m'(v) =$$

$$\int_0^\infty \mathrm{d}v\, v^{1-s} \sum_{m=-\infty}^{\infty} m^2 K_m^2(v) I_m(v) I_m'(v) = -\frac{1}{16\pi^{1/2}} \frac{\Gamma^2\left(\frac{5-s}{2}\right)\Gamma^2\left(\frac{3-s}{2}\right)}{\Gamma(4-s)} \frac{\Gamma\left(\frac{s-2}{2}\right)}{\Gamma\left(\frac{s+1}{2}\right)}. \tag{19}$$

Although the left-hand side of each integral is not initially defined for $s = -1$, the right-hand side together with the remaining s dependent factors in $\mathcal{E}_C(s)$ will eventually provide the desired extension to negative s through the existing analytic continuations of the involved functions. Then, the poles at $s = -1, -3, -5, \ldots$ in the last dividing gamma functions will give rise to zeros at these points.

Going back to $\mathcal{E}^0_{C1}(s)$, since (19) show that the two integrals in the second line of (17) have the same value,

$$\mathcal{E}^0_{C1}(s) = 0, \tag{20}$$

even before setting $s = -1$.

Formulae (14) tell us that $\mathcal{E}^1_{C1}(s)$ involves the integration of the $L^1_{m1}(\mathrm{i}v)$ function, defined by (9), (10). Therefore,

$$\mathcal{E}^1_{C1}(s) = -\frac{\hbar}{2}\frac{sa^{s-1}}{2\pi^2}\mathrm{B}\left(\frac{1}{2}, 1-\frac{s}{2}\right)\sin\left(\pi\frac{s}{2}\right)\sum_{m=-\infty}^{\infty}\int_0^\infty \mathrm{d}v\, v^{2-s} \times\left[I'_m(v)K'_m(v) - \left(1+\frac{m^2}{v^2}\right)I_m(v)K_m(v) + \frac{1}{v}(I_m(v)K_m(v))'\right]. \tag{21}$$

We multiply, again, each term in the m summation of (21) by $1 = -vW[I_m(v), K_m(v)]$, and turn the initial expression into a linear combination of integrals with summations of products of four Bessel functions. That linear combination yields an identically null result—one that is zero for any s value—by virtue of the symmetries observed in (19) under interchange of different Bessel function types (see also comment after equation (80) in [5]). As a result,

$$\mathcal{E}^1_{C1}(s) = 0. \tag{22}$$

Equation (21) admits the following reinterpretation. Taking into account the fact that I_m, K_m satisfy the modified Bessel equation, we apply partial integration to (21) omitting a 'boundary term' which vanishes for a given s range that does not include $s = -1$ yet. Doing so, we find

$$\mathcal{E}^1_{C1}(s) = -\frac{\hbar}{2}\frac{sa^{s-1}}{2\pi^2}\mathrm{B}\left(\frac{1}{2}, 1-\frac{s}{2}\right)\sin\left(\pi\frac{s}{2}\right) \times\left[\int_0^\infty \mathrm{d}v\, v^{1-s}\sum_{m=-\infty}^{\infty}(I_m(v)K_m(v))' + \frac{2}{1-s}\int_0^\infty \mathrm{d}v\, v^{2-s}\sum_{m=-\infty}^{\infty}I_m(v)K_m(v)\right]. \tag{23}$$

These integrals cannot be straightforwardly taken at $s = -1$ but, if this is ignored, we may formally put $s = -1$ and get

$$\mathcal{E}^1_{C1}(-1) \to -\frac{\hbar}{8\pi a^2}\int_0^\infty \mathrm{d}v\, v^2\sum_{m=-\infty}^{\infty}(I_m(v)K_m(v))' - \frac{\hbar}{8\pi a^2}\int_0^\infty \mathrm{d}v\, v^3\sum_{m=-\infty}^{\infty}I_m(v)K_m(v). \tag{24}$$

The first part could arguably be dismissed as a mere contact term because, from (18), it may be shown that it is local in v. (In fact it is possible to obtain $\lim_{\phi\to 0}\sum_{m=-\infty}^{\infty}(I_m(v)K_m(v))'\,\mathrm{e}^{\mathrm{i}m\phi} = -\frac{1}{v}$.) The second part of (24) cancels the bulk contribution found in [5]. (See formulae (72), (78) there and recall that the Casimir radial pressure is $P_C = \frac{1}{\pi a^2}\mathcal{E}_C$.)

Viewed in a different way, by the arguments in [13] (and references therein) all linear terms in $(\varepsilon_2 - \varepsilon_1)$ have to be removed because they are the self-energy of the electromagnetic field due to polarizable particles. By that rule, one simply must take out the linear part, regardless of its particular form. This is actually a re-statement of the physical reason for the removal of the bulk contribution.

When going on to second order in $(\varepsilon - 1)$, we take first the piece called $\mathcal{E}^{20A}_{C2}(s)$, as its calculation is most similar to that of $\mathcal{E}^0_{C1}(s), \mathcal{E}^1_{C1}(s)$. From the $\mathcal{E}^{20A}_{C2}(s)$ given in (15), the $L^{20A}_{m2}(y)$ in (9), expressions (10) with $y = \mathrm{i}v$, introducing, once more, $1 = -vW[I_m(v), K_m(v)]$, and using the same reasoning that led to (22), one finds

$$\mathcal{E}^{20A}_{C2}(s) = 0. \tag{25}$$

Now, selecting the lines in (15), which determine $\mathcal{E}^{00}_{C2}(s), \mathcal{E}^{10}_{C2}(s), \mathcal{E}^{20B}_{C2}(s), \mathcal{E}^{11}_{C2}(s)$, the parts of (9) which define $L^{00}_{m2}(y), L^{10}_{m2}(y), L^{20B}_{m2}(y), L^{11}_{m2}(y)$, the form of $\Delta^{(1,0)}_m(y)$ dictated by (10) (its square for the case of $L_{m2}^{20B}(y)$), and setting $y = \mathrm{i}v$, we obtain

$$
\begin{aligned}
\mathcal{E}_{C2}^{00}(s) &= \frac{\hbar}{2}\frac{sa^{s-1}}{4\pi^2}\mathrm{B}\left(\frac{1}{2}, -\frac{s}{2}\right)\sin\left(-\pi\frac{s}{2}\right)\int_0^\infty \mathrm{d}v\, v^{2-s}\sum_{m=-\infty}^{\infty} I_m'^{\,2}(v)K_m^2(v),\\
\mathcal{E}_{C2}^{10}(s) &= \frac{\hbar}{2}\frac{sa^{s-1}}{4\pi^2}\mathrm{B}\left(\frac{1}{2}, 1-\frac{s}{2}\right)\sin\left(\pi\frac{s}{2}\right)\int_0^\infty \mathrm{d}v\, v^{2-s}\\
&\quad\times\sum_{m=-\infty}^{\infty}\left[2I_m(v)I_m'(v)K_m(v)K_m'(v) + vI_m'^{\,2}(v)K_m(v)K_m'(v)\right.\\
&\quad\left. -\left(v+\frac{m^2}{v}\right)I_m^2(v)K_m(v)K_m'(v)\right],\\
\mathcal{E}_{C2}^{20B}(s) &= \frac{\hbar}{2}\frac{sa^{s-1}}{8\pi^2}\mathrm{B}\left(\frac{1}{2}, 2-\frac{s}{2}\right)\sin\left(-\pi\frac{s}{2}\right)\int_0^\infty \mathrm{d}v\, v^{2-s}\\
&\quad\times\sum_{m=-\infty}^{\infty}\left[I_m'^{\,2}(v)K_m^2(v) + I_m^2(v)K_m'^{\,2}(v)\right.\\
&\quad+2(1-v^2-m^2)I_m(v)I_m'(v)K_m(v)K_m'(v)\\
&\quad+v^2I_m'^{\,2}(v)K_m'^{\,2}(v) + \left(v^2+2m^2+\frac{m^4}{v^2}\right)I_m^2(v)K_m^2(v)\\
&\quad+2v\left(I_m'^{\,2}(v)K_m(v)K_m'(v) + I_m(v)I_m'(v)K_m'^{\,2}(v)\right)\\
&\quad\left.-2\left(v+\frac{m^2}{v}\right)\left(I_m(v)I_m'(v)K_m^2(v) + I_m^2(v)K_m(v)K_m'(v)\right)\right],\\
\mathcal{E}_{C2}^{11}(s) &= \frac{\hbar}{2}\frac{sa^{s-1}}{2\pi^2}\mathrm{B}\left(\frac{3}{2}, 1-\frac{s}{2}\right)\sin\left(\pi\frac{s}{2}\right)\int_0^\infty \mathrm{d}v\, v^{-s}\sum_{m=-\infty}^{\infty} m^2I_m^2(v)K_m^2(v).
\end{aligned}
\tag{26}
$$

The outcome of replacing the results (19) into (26) and expanding in $(s+1)$ is

$$
\mathcal{E}_{C2}^{00}(s) + \mathcal{E}_{C2}^{10}(s) + \mathcal{E}_{C2}^{20B}(s) + \mathcal{E}_{C2}^{11}(s) = \frac{\hbar}{a^2}\widehat{\mathcal{E}}(s+1) + \mathcal{O}((s+1)^2), \tag{27}
$$

with $\widehat{\mathcal{E}} = \frac{23}{5760\pi}$. Actually, each term vanishes at $s = -1$. Formulae (25) and (27) make evident that

$$
\lim_{s\to -1}\mathcal{E}_{C2}(s) = 0, \tag{28}
$$

i.e., the $(\varepsilon - 1)^2$ contribution to the Casimir energy per unit length in the dilute-dielectric approximation is zero, as we wished to prove.

Employing a regularization which analytically continues the vacuum energy as a function of the eigenmode power, we have found a pure Casimir term (in the sense of [2]) that is seen to vanish through the order of $(\varepsilon - 1)^2$. Remarkably, for the analogous problem with light velocity conservation condition [1, 12] the result is null through the order of $\xi^2 \equiv \left(\frac{\varepsilon_1-\varepsilon_2}{\varepsilon_1+\varepsilon_2}\right)^2$. In fact, we have applied a form of zeta function regularization, whose links to other techniques have been studied in e.g. [14]. The sight of (19) makes us evoke the words of [15] and proclaim that a forest of gamma functions has grown out of an analytic continuation.

A divergence at third order in $(\varepsilon - 1)$ introduces an unavoidable ambiguity [4] (for further discussions on divergences see [16].) No universal agreement exists on the interpretation of the physical significance of such infinities, as commented in [15]. The nature of a third order divergence, viewed as a weak-coupling limit, has been considered in [17].

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